7. L. P. Menakhin, E. K. Eroshchenkov, and K. N. Ul'yanov, "Current instability in molecular gases," Zh. Tekh. Fiz., 45, No. 6 (1975).
8. O. B. Evdokimov, V. V. Kremnev, G. A. Mesyats, and V. B. Ponomarev, "Field distribution in a volume gas discharge triggered by a beam of fast electrons," Zh. Tekh. Fiz., 43, No. 11 (1973).

EQUILIBRIUM AND STABILITY OF THE SEPARATION SURFACE BETWEEN A LIQUID DIELECTRIC AND A PERFECTLY CONDUCTING LIQUID

A. I. Zhakin

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We consider a stationary container (Fig. 1) completely filled with two perfectly immiscible liquids, one of which is a perfect conductor, and the other a dielectric with a dielectric constant ε. We assume that the container walls are perfect conductors, both liquids are at the same temperature, and the liquid–liquid interface has no points in common with the container walls.

We introduce the following notation: $\Omega_1(\Omega_2)$ is the region occupied by the conductor (dielectric); $S_1(S_2)$ is the container wall adjoining the conductor (dielectric); Γ is the surface of separation of the liquids; $\mathbf{n}$ is a unit vector normal to Γ, directed into the region Ω_1; $\mathbf{n}_i$ is a unit vector normal to S_i and directed into the region $\Omega = \Omega_1 + \Omega_2$; ρ_i is the density of the conductor (dielectric); σ is the surface tension in the liquid–liquid interface; φ is the electric potential; $\mathbf{r}$ is the radius-vector to a point; and V_i is the volume of region Ω_i.

We assume that the electric field results from a potential difference U between S_1 and S_2 and that the external forces have a potential Π_i in Ω_i (i = 1, 2).

1. Condition for Equilibrium of Liquids in a Container

In deriving the equilibrium conditions we start from the variational principle that the potential energy has a stationary value. The potential energy is

$$W = \sigma \int_\Gamma d\Gamma + \sum_{i=1}^{2} \int_{\Omega_i} \Pi_i d\Omega - \frac{\varepsilon}{8\pi} \int_\Gamma \varphi \frac{\partial \varphi}{\partial n} d\Gamma + \text{const}. \quad (1.1)$$

Let $\mathbf{h}(\mathbf{r})$ be the displacement of a liquid particle. We assume that $\mathbf{h}(\mathbf{r})$ is a twice continuously differentiable function, continuous in Ω, having no normal component on S and a continuous normal component on Γ,

$$\operatorname{div} \mathbf{h}(\mathbf{r}) = 0 \qquad (\mathbf{r} \in \Omega); \quad (1.2)$$

$$\mathbf{h}(\mathbf{r})\mathbf{n}_i = 0 \qquad (i = 1, 2;\ \mathbf{r} \in S = S_1 + S_2); \quad (1.3)$$

$$\lim_{\mathbf{r}_1 \to \mathbf{r}} \mathbf{h}(\mathbf{r}_1)\mathbf{n} = \lim_{\mathbf{r}_2 \to \mathbf{r}} \mathbf{h}(\mathbf{r}_2)\mathbf{n} \quad (\mathbf{r} \in \Gamma,\ \mathbf{r}_1 \in \Omega_1, \quad \mathbf{r}_2 \in \Omega_2). \quad (1.4)$$

We assume that U = const in virtual displacements of a liquid particle. This is possible only if there is an external energy source [1].

Using the formulas for the variations of the area of a surface and a unit vector normal to a surface [2] we obtain an expression for the first variation of the potential energy in the form

$$\delta W = \int_\Gamma \left(-2\sigma H + \frac{\varepsilon}{8\pi} \left(\frac{\partial \varphi}{\partial n} \right)^2 \right) N d\Gamma + \sum_{i=1}^{2} \int_{\Omega_i} \nabla \Pi_i \mathbf{h} d\Omega,$$

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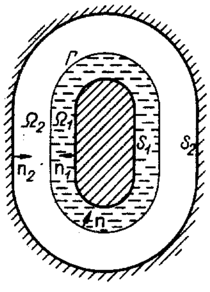

Fig. 1

where H is the mean curvature of the surface Γ, and N = **h(r)n**. By using the rule for Lagrangian undetermined multipliers [for condition (1.2)] and the Gauss theorem, the necessary condition for (1.1) to be stationary is the vanishing of the expression

$$\delta W_* = \int\limits_{\Gamma} \left(-2\sigma H - [p] + \frac{\varepsilon}{8\pi}\left(\frac{\partial\varphi}{\partial n}\right)^2\right) N d\Gamma + \sum_{i=1}^{2} \int\limits_{\Omega_i} \nabla(\Pi_i + p_i)\, \mathbf{h} d\Omega \tag{1.5}$$

for all h(r) which satisfy Eqs. (1.3) and (1.4). In Eq. (1.5) $[p] = p_1 - p_2$ is the discontinuity in the quantity p in passing through the surface Γ. Since **h(r)** is quite arbitrary, the relations

$$-2\sigma H - [p] + (\varepsilon/8\pi)(\partial\varphi/\partial n)^2 = 0 \text{ on } \Gamma; \tag{1.6}$$

$$\Pi_i + p_i = c_i \text{ in } \Omega_i \tag{1.7}$$

must be satisfied, where the c_i (i = 1, 2) are constants. Using the equations of electrostatics and (1.6) and (1.7) we obtain the following equations and boundary conditions describing the equilibrium state of the system:

$$-2\sigma H + [\Pi] + (\varepsilon/8\pi)(\partial\varphi/\partial n)^2 + c = 0 \text{ on } \Gamma; \tag{1.8}$$

$$\int\limits_{\Omega_i} d\Omega = V_i \quad (i = 1, 2); \tag{1.9}$$

$$\Delta\varphi = 0 \text{ in } \Omega_2, \ \varphi|_{S_2} = 0, \ \varphi|_{\Gamma} = U, \tag{1.10}$$

where c is a constant.

2. Stability of the Equilibrium State of the Liquids

By the stability of the equilibrium state of the liquids we understand the stability of the equilibrium shapes of the surface of separation Γ [3]. We assume that the principle of minimum potential energy holds [3], and therefore, except in special cases, the stability of the equilibrium state of the system can be determined from the sign of the second variation of the potential energy.

By varying the first variation (1.5) and using the fact that Eqs. (1.8)-(1.10) hold in the equilibrium state and that $\Pi_i(\mathbf{r})$ (i = 1, 2; $\mathbf{r} \in \Omega$) is a given local function, we obtain

$$\sigma^{-1}\delta^2 W_* = \int\limits_{\Gamma} \left(-\Delta_\Gamma N + aN + \frac{\varepsilon}{8\pi}\frac{\partial\varphi}{\partial n}\frac{\partial\psi}{\partial n}\right) N d\Gamma, \tag{2.1}$$

where

$$a = \sigma^{-1}\frac{\partial[\Pi]}{\partial n} - 4H^2 + 2K + \frac{1}{2\pi\sigma} H \left(\frac{\partial\varphi}{\partial n}\right)^2;$$

ψ is the local perturbation of the electric potential φ; Δ_Γ is the Laplace—Beltrami operator [2]; and K is the Gaussian curvature. The function N(**r**) ($\mathbf{r} \in \Gamma$) must satisfy the conservation of volume condition

$$\int\limits_{\Gamma} N d\Gamma = 0, \tag{2.2}$$

which follows from (1.2). The function ψ(**r**) ($\mathbf{r} \in \Omega_2$) must satisfy the following boundary-value problem:

$$\Delta\psi = 0 \text{ in } \Omega_2; \tag{2.3}$$

$$\psi|_{S_2} = 0,\ \psi|_\Gamma = -\frac{\partial\varphi}{\partial n} N. \tag{2.4}$$

We introduce the normalization condition for N(r)

$$\int_\Gamma N^2 d\Gamma = 1; \tag{2.5}$$

then the equilibrium state described by Eqs. (1.8)-(1.10) is stable if the minimum of the quadratic functional (2.1) for conditions (2.2)-(2.5) is positive, and unstable otherwise.

We note that if the quadratic functional (2.1) is bounded below on a set of twice continuously differentiable functions N(r) satisfying conditions (2.2) and (2.5), its minimum on this set of functions is equal to the smallest eigenvalue $*$ of the following boundary-value problem [4]:

$$-\Delta_\Gamma N + aN + \frac{\varepsilon}{4\pi}\frac{\partial\varphi}{\partial n}\frac{\partial\psi}{\partial n} + m = \lambda N \text{ on } \Gamma; \tag{2.6}$$

$$\int_\Gamma N d\Gamma = 0,\ \int_\Gamma N^2 d\Gamma = 1; \tag{2.7}$$

$$\Delta\psi = 0,\ \psi|_{S_2} = 0,\ \psi|_\Gamma = -\frac{\partial\varphi}{\partial n} N, \tag{2.8}$$

where m is a constant.

3. Solution of Certain Problems

In investigating the stability of the equilibrium state of the problems considered below we use Eqs. (2.6)-(2.8), since for the cases under consideration it can be shown that the quadratic functional (2.1) is bounded below.

1. We consider the two-dimensional problem of the stability of a system consisting of two conductors separated by a dielectric (Fig. 2). The liquids are contained between two parallel electrodes with a constant potential difference U between them. We introduce the coordinate system shown in Fig. 2 and the subscripts 1, 2, and 3 to denote, respectively, the upper and lower conductors and the dielectric. The gravitational field is $\mathbf{g} = -g\mathbf{e}_y$, where $\mathbf{e}_y$ is a unit vector along the Oy axis.

We introduce the following notation: Ω is the region occupied by the dielectric; $\Gamma_1(\Gamma_2)$ is the surface of separation between the upper (lower) conductor and the dielectric; d is the thickness of the dielectric layer; $\mathbf{n}$ is a unit vector normal to $\Gamma = \Gamma_1 + \Gamma_2$, directed into the region occupied by the conductors; σ_i is the surface tension in Γ_i; and $y = f_i(x)$ is the surface of separation Γ_i $(i = 1, 2)$.

In the present case condition (1.8) on Γ_i is not changed, but (1.9) is replaced by

$$\lim_{T\to\infty}\frac{1}{T}\int_{-T}^{T} f_i(x)\,dx = 0 \quad (i = 1, 2).$$

We take conditions (1.10) in the form

$$\Delta\varphi = 0 \text{ in } \Omega,\ \varphi|_{\Gamma_1} = 0,\ \varphi|_{\Gamma_2} = U.$$

It is clear that the equilibrium state of the system is described by the equations

$$\Gamma_1 : f_1 \equiv 0,\ \Gamma_2 : f_2 \equiv d,\ \varphi = Ey,\ E = U/d.$$

The system is stable if the minimum eigenvalue λ_* of the following boundary-value problem is positive:

$$-\sigma_1\Delta_{\Gamma_1} N_1 + a_1 N_1 + \frac{\varepsilon}{4\pi}\frac{\partial\varphi}{\partial n}\frac{\partial\psi}{\partial n} + m_1 = \lambda N_1 \text{ on } \Gamma_1; \tag{3.1}$$

$$-\sigma_2\Delta_{\Gamma_2} N_2 + a_2 N_2 + \frac{\varepsilon}{4\pi}\frac{\partial\varphi}{\partial n}\frac{\partial\psi}{\partial n} + m_2 = \lambda N_2 \text{ on } \Gamma_2; \tag{3.2}$$

$$\lim_{T\to\infty}\frac{1}{T}\int_{-T}^{T} N_i(x)\,dx = 0,\quad \lim_{T\to\infty}\frac{1}{T}\int_{-T}^{T}\left(N_1^2(x) + N_2^2(x)\right)dx = 1; \tag{3.3}$$

$$\Delta\psi = 0,\ \psi|_{\Gamma_1} = -\frac{\partial\varphi}{\partial n} N_1,\ \psi|_{\Gamma_2} = -\frac{\partial\varphi}{\partial n} N_2 \quad (i = 1, 2), \tag{3.4}$$

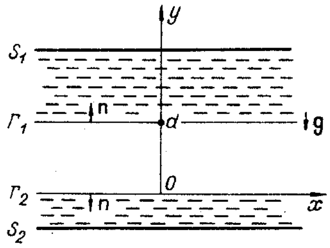

Fig. 2

where $a_1 = g(\rho_2 - \rho_1)$, $a_2 = g(\rho_3 - \rho_2)$, and m_1 and m_2 are constants. The eigenfunctions of the boundary-value problem (3.1)-(3.4) are sin kx and cos kx, where k is the wave number, and therefore we seek the solution in the form

$$N_1 = A_1 \sin kx,\ N_2 = A_2 \sin kx. \tag{3.5}$$

Then the perturbation of the electric potential has the form

$$\psi = -(E/\text{sh}\, kd) \sin kx(\text{sh}\, k(y-d) + \text{sh}\, ky). \tag{3.6}$$

Substituting (3.5) and (3.6) into (3.1) and (3.2) we find a homogeneous linear system for the coefficients A_1 and A_2. Equating the determinant of this system to zero we obtain an expression for the eigenvalue of the problem as a function of k,

$$\lambda = \lambda(k) = (1/2)(F \pm \sqrt{F^2 - 4B}), \tag{3.7}$$

where

$$F = \sigma_1 k^2 + a_1 + \sigma_2 k^2 + a_2 - (\varepsilon/2\pi)E^2 k \text{ cth } kd;$$
$$B = (\sigma_1 k^2 + a_1)(\sigma_2 k^2 + a_2) - (\varepsilon/4\pi)E^2 k \text{ cth } kd\, (\sigma_1 k^2 + a_1 + \sigma_2 k^2 + a_2).$$

It is clear that Eq. (3.7) is minimum for a certain $k = k_*$. The critical value of the electric field intensity is found from the condition

$$\lambda_* = \lambda(k_*) = 0.$$

Let us analyze the limiting cases $d \to 0$ and $d \to \infty$ for E = const. It can be seen that if the minus sign is taken in the expression in parentheses in (3.7), $\lambda(k) \to -\infty$ as $d \to 0$ for any fixed k; i.e., the closer together the conducting planes the less stable the system. Letting $d \to \infty$ we find

$$\lambda = (1/2)[\sigma_1 k^2 + a_1 + \sigma_2 k^2 + a_2 - (\varepsilon/2\pi)E^2 k \pm (\sigma_1 k^2 + a_1 - \sigma_2 k^2 - a_2)],$$

for which

$$\lambda^* = \min_k \lambda = \min\{\min_k \lambda_1,\ \min_k \lambda_2\}, \tag{3.8}$$

where

$$\lambda_i = \sigma_i k^2 + a_i - (\varepsilon/4\pi)\ E^2 k (i = 1, 2). \tag{3.9}$$

It follows from (3.8) that as the distance between the conducting planes is increased, their effect on one another decreases, and in the limit the problem reduces to that of the stability of the plane boundary of separation between two semiinfinite regions, one of which is a perfect conductor and the other a dielectric, and the electric field at infinity is constant and perpendicular to the surface of separation. This problem was treated in [5]. The expressions for the critical value of the electric field obtained in [5] and from (3.9) agree:

$$E_*^2 = \frac{8\pi}{\varepsilon} \sqrt{g[\rho]\sigma},$$

where $[\rho]$ is the discontinuity in density in passing through the surface of separation, and σ is the surface tension in the surface of separation.

2. The two-dimensional problem of the stability of a system consisting of two dielectrics separated by a perfect conductor and located between two parallel electrodes can be treated in a similar way. The minimum eigenvalue of this problem is given by (3.8), where it is necessary to set

$$\lambda_i = \sigma_i k^2 + a_i - \frac{\varepsilon_i}{4\pi} E_i^2 k;$$

here ε_i is the dielectric constant of the i-th dielectric, and E_i is the electric field intensity in the i-th dielectric (i - 1, 2).

In conclusion, we note that this method can be used to investigate the stability of the equilibrium state of a system consisting of several alternating layers of liquid dielectrics and conductors.

3. Let us consider the stability of a spherical drop of radius R carrying a charge Q and having negligible weight ($|g[\rho]R^2| \ll \sigma$). We introduce the spherical coordinates r, α_1, α_2, where r is the distance from the origin of the center of the drop; and α_1 and α_2 are, respectively, the azimuthal and polar angles. Clearly, the equilibrium state is described by the equations

$$\Gamma : r \equiv R, \qquad \varphi = -UR/r,$$

where $U = Q/\varepsilon R$.

The boundary-value problem whose minimum eigenvalue is sought will have a form similar to (2.6)-(2.8), except that in the present case it is necessary to require that the perturbation of the electric potential ψ be bounded at infinity. The eigenfunctions of the boundary-value problem have the form

$$N_{ij} = A_{ij} \cos i\alpha_1 P_i^j (\cos \alpha_2) \quad (3.10)$$
$$(i = 0,1,2,\ldots;\ j = 0,\ \pm 1,\ \pm 2,\ \ldots,\ \pm i),$$

where the $P_i^j(\cos \alpha_2)$ are associated Legendre functions. The perturbation of the electric potential is given by

$$\psi_{ij} = A_{ij} E \left(\frac{R}{r}\right)^i \cos j\alpha_1 P_i^j (\cos \alpha_2), \quad (3.11)$$

where $E = U/R$. Substituting (3.10) and (3.11) into (2.6) we have for the case under consideration

$$\lambda_i = \frac{i(i+1)}{R^2} - \frac{2}{R^2} - (i-1)\frac{Q^2}{4\pi\varepsilon\sigma R^5}$$
$$(i = 0,\ 1,\ 2,\ \ldots).$$

We consider perturbations which leave the center of mass of the drop stationary. Then the values of λ_i for i = 0 and i = 1 are eliminated from consideration, and the condition for stability of a stationary charged drop takes the form

$$Q^2 < 16\pi\varepsilon\sigma R^3.$$

The charge $Q = \sqrt{16\varepsilon\sigma\pi R^3}$ is called the limiting Rayleigh charge [7]. By considering the analogous problem for the plane case the stability criterion can be obtained in the form

$$Q^2 < 3\pi\varepsilon\sigma R,$$

where Q is the charge per unit length of a charged filament.

LITERATURE CITED

1. I. E. Tamm, Principles of the Theory of Electricity [in Russian], Nauka, Moscow (1976).
2. V. Blyashke, Differential Geometry [in Russian], ONTI, Moscow (1935).
3. N. N. Moiseev and V. V. Rumyantsev, Dynamics of a Body with Cavities Containing a Liquid [in Russian], Nauka, Moscow (1965).
4. S. G. Mikhlin, Variational Methods in Mathematical Physics, Pergamon, New York (1964).
5. Ya. I. Frenkel', "On Tonk's theory of the dislocation of the surface of a liquid by an electrostatic field in vacuum," Zh. Éksp. Teor. Fiz., 6, No. 4 (1936).
6. L. D. Landau and E. M. Lifshits, Electrodynamics of a Continuous Medium, Pergamon, New York (1960).
7. J. R. Melcher, "Electrohydrodynamics," Magn. Gidrodin., No. 2 (1974).